Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, A, B and C. All sections are compulsory. Questions in each section are of different types.

2. Section – A contains a total of 30 Multiple Choice Questions (MCQ). Each MCQ type question has four choices out of which only one choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. Section – B contains a total of 10 Multiple Select Questions (MSQ). Each MSQ type question is similar to MCQ but with a difference that there may be one or more than one choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. Section – C contains a total of 20 Numerical Answer Type (NAT) questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In Section – A (MCQ), wrong answer will result in NEGATIVE marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In Section – B (MSQ), there is NO NEGATIVE and NO PARTIAL marking provisions. There is NO NEGATIVE marking in Section – C (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are NOT allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.
**Notation**

\( \mathbb{N} \) \hspace{1cm} set of all natural numbers \( 1, 2, 3, \cdots \)  
\( \mathbb{R} \) \hspace{1cm} set of all real numbers  
\( M_{m \times n}(\mathbb{R}) \) \hspace{1cm} real vector space of all matrices of size \( m \times n \) with entries in \( \mathbb{R} \)  
\( \emptyset \) \hspace{1cm} empty set  
\( X \setminus Y \) \hspace{1cm} set of all elements from the set \( X \) which are not in the set \( Y \)  
\( \mathbb{Z}_n \) \hspace{1cm} group of all congruence classes of integers modulo \( n \)  
\( \hat{i}, \hat{j}, \hat{k} \) \hspace{1cm} unit vectors having the directions of the positive \( x, y \) and \( z \) axes of a three dimensional rectangular coordinate system, respectively  
\( S_n \) \hspace{1cm} group of all permutations of the set \( \{1, 2, 3, \cdots, n\} \)  
\( \ln \) \hspace{1cm} logarithm to the base \( e \)  
\( \log \) \hspace{1cm} logarithm to the base 10  
\( \nabla \) \hspace{1cm} \( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \)  
\( \det(M) \) \hspace{1cm} determinant of a square matrix \( M \)
SECTION – A
MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 Let \( a_1 = b_1 = 0 \), and for each \( n \geq 2 \), let \( a_n \) and \( b_n \) be real numbers given by

\[
a_n = \sum_{m=2}^{n} \frac{(-1)^m m}{(\log(m))^m} \quad \text{and} \quad b_n = \sum_{m=2}^{n} \frac{1}{(\log(m))^m}.
\]

Then which one of the following is TRUE about the sequences \( \{a_n\} \) and \( \{b_n\} \)?

(A) Both \( \{a_n\} \) and \( \{b_n\} \) are divergent
(B) \( \{a_n\} \) is convergent and \( \{b_n\} \) is divergent
(C) \( \{a_n\} \) is divergent and \( \{b_n\} \) is convergent
(D) Both \( \{a_n\} \) and \( \{b_n\} \) are convergent

Q.2 Let \( T \in M_{m \times n}(\mathbb{R}) \). Let \( V \) be the subspace of \( M_{n \times p}(\mathbb{R}) \) defined by

\[
V = \{ X \in M_{n \times p}(\mathbb{R}) : TX = 0 \}.
\]

Then the dimension of \( V \) is

(A) \( pn - \text{rank}(T) \)  
(B) \( mn - p \text{ rank}(T) \)  
(C) \( p(m - \text{ rank}(T)) \)  
(D) \( p(n - \text{ rank}(T)) \)

Q.3 Let \( g : \mathbb{R} \to \mathbb{R} \) be a twice differentiable function. Define \( f : \mathbb{R}^3 \to \mathbb{R} \) by

\[
f(x, y, z) = g(x^2 + y^2 - 2z^2).
\]

Then \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \) is equal to

(A) \( 4(x^2 + y^2 - 4z^2) g''(x^2 + y^2 - 2z^2) \)  
(B) \( 4(x^2 + y^2 + 4z^2) g''(x^2 + y^2 - 2z^2) \)  
(C) \( 4(x^2 + y^2 - 2z^2) g''(x^2 + y^2 - 2z^2) \)  
(D) \( 4(x^2 + y^2 + 4z^2) g''(x^2 + y^2 - 2z^2) + 8g'(x^2 + y^2 - 2z^2) \)

Q.4 Let \( \{a_n\}_{n=0}^\infty \) and \( \{b_n\}_{n=0}^\infty \) be sequences of positive real numbers such that \( na_n < b_n < n^2 a_n \) for all \( n \geq 2 \). If the radius of convergence of the power series \( \sum_{n=0}^\infty a_n x^n \) is 4, then the power series \( \sum_{n=0}^\infty b_n x^n \)

(A) converges for all \( x \) with \( |x| < 2 \) 
(B) converges for all \( x \) with \( |x| > 2 \) 
(C) does not converge for any \( x \) with \( |x| > 2 \) 
(D) does not converge for any \( x \) with \( |x| < 2 \)

Q.5 Let \( S \) be the set of all limit points of the set \( \left\{ \frac{n^2 + \sqrt{2}}{n} : n \in \mathbb{N} \right\} \). Let \( \mathbb{Q}_+ \) be the set of all positive rational numbers. Then

(A) \( \mathbb{Q}_+ \subseteq S \)  
(B) \( S \subseteq \mathbb{Q}_+ \)  
(C) \( S \cap (\mathbb{R} \setminus \mathbb{Q}_+) \neq \emptyset \)  
(D) \( S \cap \mathbb{Q}_+ \neq \emptyset \)
Q.6 If $x^h y^k$ is an integrating factor of the differential equation
\[ y(1 + xy) \, dx + x(1 - xy) \, dy = 0, \]
then the ordered pair $(h, k)$ is equal to
\(\text{(A)} \ (-2, -2) \quad \text{(B)} \ (-2, -1) \quad \text{(C)} \ (-1, -2) \quad \text{(D)} \ (-1, -1)\)

Q.7 If $y(x) = \lambda e^{2x} + e^{\beta x}$, $\beta \neq 2$, is a solution of the differential equation
\[ \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \]
satisfying $\frac{dy}{dx}(0) = 5$, then $y(0)$ is equal to
\(\text{(A)} \ 1 \quad \text{(B)} \ 4 \quad \text{(C)} \ 5 \quad \text{(D)} \ 9\)

Q.8 The equation of the tangent plane to the surface $x^2z + \sqrt{8 - x^2 - y^4} = 6$ at the point $(2, 0, 1)$ is
\(\text{(A)} \ 2x + z = 5 \quad \text{(B)} \ 3x + 4z = 10 \quad \text{(C)} \ 3x - z = 10 \quad \text{(D)} \ 7x - 4z = 10\)

Q.9 The value of the integral
\[ \int_{y=0}^{1} \int_{x=0}^{1-y^2} y \sin(\pi(1 - x)^2) \, dx \, dy \]
is
\(\text{(A)} \ \frac{1}{2\pi} \quad \text{(B)} \ 2\pi \quad \text{(C)} \ \frac{\pi}{2} \quad \text{(D)} \ \frac{2}{\pi}\)

Q.10 The area of the surface generated by rotating the curve $x = y^3$, $0 \leq y \leq 1$, about the $y$-axis, is
\(\text{(A)} \ \frac{\pi}{27} 10^{3/2} \quad \text{(B)} \ \frac{4\pi}{3} (10^{3/2} - 1) \quad \text{(C)} \ \frac{\pi}{27} (10^{3/2} - 1) \quad \text{(D)} \ \frac{4\pi}{3} 10^{3/2}\)

Q.11 – Q.30 carry two marks each.

Q.11 Let $H$ and $K$ be subgroups of $\mathbb{Z}_{444}$. If the order of $H$ is 24 and the order of $K$ is 36, then the order of the subgroup $H \cap K$ is
\(\text{(A)} \ 3 \quad \text{(B)} \ 4 \quad \text{(C)} \ 6 \quad \text{(D)} \ 12\)

Q.12 Let $P$ be a $4 \times 4$ matrix with entries from the set of rational numbers. If $\sqrt{2} + i$, with $i = \sqrt{-1}$, is a root of the characteristic polynomial of $P$ and $I$ is the $4 \times 4$ identity matrix, then
\(\text{(A)} \ P^4 = 4P^2 + 9I \quad \text{(B)} \ P^4 = 4P^2 - 9I \quad \text{(C)} \ P^4 = 2P^2 - 9I \quad \text{(D)} \ P^4 = 2P^2 + 9I\)
Q.13 The set \( \left\{ \frac{x}{1+x} : -1 < x < 1 \right\} \), as a subset of \( \mathbb{R} \), is

(A) connected and compact
(B) connected but not compact
(C) not connected but compact
(D) neither connected nor compact

Q.14 The set \( \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\} \cup \{0\} \), as a subset of \( \mathbb{R} \), is

(A) compact and open
(B) compact but not open
(C) not compact but open
(D) neither compact nor open

Q.15 For \(-1 < x < 1\), the sum of the power series \( 1 + \sum_{n=2}^{\infty} (-1)^{n-1} n^2 x^{n-1} \) is

(A) \( \frac{1-x}{(1+x)^3} \)
(B) \( \frac{1+x^2}{(1+x)^4} \)
(C) \( \frac{1-x}{(1+x)^2} \)
(D) \( \frac{1+x^2}{(1+x)^3} \)

Q.16 Let \( f(x) = (\ln x)^2, x > 0. \) Then

(A) \( \lim_{x \to \infty} \frac{f(x)}{x} \) does not exist
(B) \( \lim_{x \to \infty} f'(x) = 2 \)
(C) \( \lim_{x \to \infty} (f(x+1) - f(x)) = 0 \)
(D) \( \lim_{x \to \infty} (f(x+1) - f(x)) \) does not exist

Q.17 Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that \( f'(x) > f(x) \) for all \( x \in \mathbb{R} \), and \( f(0) = 1. \) Then \( f(1) \) lies in the interval

(A) \((0, e^{-1})\)
(B) \((e^{-1}, \sqrt{e})\)
(C) \((\sqrt{e}, e)\)
(D) \((e, \infty)\)

Q.18 For which one of the following values of \( k \), the equation

\[ 2x^3 + 3x^2 - 12x - k = 0 \]

has three distinct real roots?

(A) 16
(B) 20
(C) 26
(D) 31

Q.19 Which one of the following series is divergent?

(A) \( \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{1}{n} \)
(B) \( \sum_{n=1}^{\infty} \frac{1}{n} \log n \)
(C) \( \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n} \)
(D) \( \sum_{n=1}^{\infty} \frac{1}{n} \tan \frac{1}{n} \)
Q.20 Let \( S \) be the family of orthogonal trajectories of the family of curves
\[
2x^2 + y^2 = k, \text{ for } k \in \mathbb{R} \text{ and } k > 0.
\]
If \( C \in S \) and \( C \) passes through the point \((1, 2)\), then \( C \) also passes through
(A) \((4, -\sqrt{2})\) \hspace{1cm} (B) \((2, -4)\) \hspace{1cm} (C) \((2, 2\sqrt{2})\) \hspace{1cm} (D) \((4, 2\sqrt{2})\)

Q.21 Let \( x, x + e^x \) and \( 1 + x + e^x \) be solutions of a linear second order ordinary differential equation with constant coefficients. If \( y(x) \) is the solution of the same equation satisfying \( y(0) = 3 \) and \( y'(0) = 4 \), then \( y(1) \) is equal to
(A) \( e + 1 \) \hspace{1cm} (B) \( 2e + 3 \) \hspace{1cm} (C) \( 3e + 2 \) \hspace{1cm} (D) \( 3e + 1 \)

Q.22 The function
\[
f(x, y) = x^3 + 2xy + y^3
\]
has a saddle point at
(A) \((0, 0)\) \hspace{1cm} (B) \((-\frac{2}{3}, -\frac{2}{3})\) \hspace{1cm} (C) \((-\frac{3}{2}, -\frac{3}{2})\) \hspace{1cm} (D) \((-1, -1)\)

Q.23 The area of the part of the surface of the paraboloid \( x^2 + y^2 + z = 8 \) lying inside the cylinder \( x^2 + y^2 = 4 \) is
(A) \( \frac{\pi}{2} (17^{3/2} - 1) \) \hspace{1cm} (B) \( \pi (17^{3/2} - 1) \) \hspace{1cm} (C) \( \frac{\pi}{6} (17^{3/2} - 1) \) \hspace{1cm} (D) \( \frac{\pi}{3} (17^{3/2} - 1) \)

Q.24 Let \( C \) be the circle \((x - 1)^2 + y^2 = 1\), oriented counter clockwise. Then the value of the line integral
\[
\oint_{C} -\frac{4}{3}xy^3 \, dx + x^4 \, dy
\]
is
(A) \( 6\pi \) \hspace{1cm} (B) \( 8\pi \) \hspace{1cm} (C) \( 12\pi \) \hspace{1cm} (D) \( 14\pi \)

Q.25 Let \( \vec{F}(x, y, z) = 2y \, \vec{i} + x^2 \, \vec{j} + xy \, \vec{k} \) and let \( C \) be the curve of intersection of the plane \( x + y + z = 1 \) and the cylinder \( x^2 + y^2 = 1 \). Then the value of
\[
\oint_{C} \vec{F} \cdot d\vec{r}
\]
is
(A) \( \pi \) \hspace{1cm} (B) \( \frac{3\pi}{2} \) \hspace{1cm} (C) \( 2\pi \) \hspace{1cm} (D) \( 3\pi \)
Q.26 The tangent line to the curve of intersection of the surface $x^2 + y^2 - z = 0$ and the plane $x + z = 3$ at the point $(1, 1, 2)$ passes through

(A) $(-1, -2, 4)$  
(B) $(-1, 4, 4)$  
(C) $(3, 4, 4)$  
(D) $(-1, 4, 0)$

Q.27 The set of eigenvalues of which one of the following matrices is NOT equal to the set of eigenvalues of \[
\begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix}
\]?

(A) \[
\begin{pmatrix}
1 & 4 \\
2 & 3
\end{pmatrix}
\]  
(B) \[
\begin{pmatrix}
3 & 2 \\
4 & 1
\end{pmatrix}
\]  
(C) \[
\begin{pmatrix}
3 & 4 \\
2 & 1
\end{pmatrix}
\]  
(D) \[
\begin{pmatrix}
2 & 3 \\
1 & 4
\end{pmatrix}
\]

Q.28 Let \{a_n\} be a sequence of positive real numbers. The series $\sum_{n=1}^{\infty} a_n$ converges if the series

(A) $\sum_{n=1}^{\infty} a_n^2$ converges 
(B) $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ converges 
(C) $\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n}$ converges 
(D) $\sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}}$ converges

Q.29 For $\beta \in \mathbb{R}$, define

$$f(x, y) = \begin{cases} 
  x^2 |x|^{\beta} y & x \neq 0, \\
  x^4 + y^2, & x = 0.
\end{cases}$$

Then, at $(0, 0)$, the function $f$ is

(A) continuous for $\beta = 0$ 
(B) continuous for $\beta > 0$ 
(C) not differentiable for any $\beta$ 
(D) continuous for $\beta < 0$

Q.30 Let \{a_n\} be a sequence of positive real numbers such that

$$a_1 = 1, \quad a_{n+1}^2 - 2a_n a_{n+1} - a_n = 0$$

for all $n \geq 1$.

Then the sum of the series $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ lies in the interval

(A) $(1, 2]$ 
(B) $(2, 3]$ 
(C) $(3, 4]$ 
(D) $(4, 5]$
SECTION - B
MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

**Q.31** Let $G$ be a noncyclic group of order 4. Consider the statements I and II:

I. There is NO injective (one-one) homomorphism from $G$ to $\mathbb{Z}_8$
II. There is NO surjective (onto) homomorphism from $\mathbb{Z}_8$ to $G$

Then

(A) I is true
(B) I is false
(C) II is true
(D) II is false

**Q.32** Let $G$ be a nonabelian group, $y \in G$, and let the maps $f, g, h$ from $G$ to itself be defined by

$$f(x) = yxy^{-1}, \quad g(x) = x^{-1} \quad \text{and} \quad h = g \circ g.$$

Then

(A) $g$ and $h$ are homomorphisms and $f$ is not a homomorphism
(B) $h$ is a homomorphism and $g$ is not a homomorphism
(C) $f$ is a homomorphism and $g$ is not a homomorphism
(D) $f, g$ and $h$ are homomorphisms

**Q.33** Let $S$ and $T$ be linear transformations from a finite dimensional vector space $V$ to itself such that $S(T(\nu)) = 0$ for all $\nu \in V$. Then

(A) $\text{rank}(T) \geq \text{nullity}(S)$
(B) $\text{rank}(S) \geq \text{nullity}(T)$
(C) $\text{rank}(T) \leq \text{nullity}(S)$
(D) $\text{rank}(S) \leq \text{nullity}(T)$

**Q.34** Let $\vec{F}$ and $\vec{G}$ be differentiable vector fields and let $g$ be a differentiable scalar function. Then

(A) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot \nabla \times \vec{F} - \vec{F} \cdot \nabla \times \vec{G}$
(B) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot \nabla \times \vec{F} + \vec{F} \cdot \nabla \times \vec{G}$
(C) $\nabla \cdot (g \vec{F}) = g\nabla \cdot \vec{F} - \nabla g \cdot \vec{F}$
(D) $\nabla \cdot (g \vec{F}) = g\nabla \cdot \vec{F} + \nabla g \cdot \vec{F}$

**Q.35** Consider the intervals $S = (0, 2]$ and $T = [1, 3)$. Let $S^*$ and $T^*$ be the sets of interior points of $S$ and $T$, respectively. Then the set of interior points of $S \setminus T^*$ is equal to

(A) $S \setminus T^*$
(B) $S \setminus T$
(C) $S^* \setminus T^*$
(D) $S^* \setminus T$

**Q.36** Let $\{a_n\}$ be the sequence given by

$$a_n = \max \left\{ \sin \left( \frac{n\pi}{3} \right), \cos \left( \frac{n\pi}{3} \right) \right\}, \quad n \geq 1.$$

Then which of the following statements is/are TRUE about the subsequences $\{a_{6n-1}\}$ and $\{a_{6n+4}\}$?

(A) Both the subsequences are convergent
(B) Only one of the subsequences is convergent
(C) $\{a_{6n-1}\}$ converges to $-\frac{1}{2}$
(D) $\{a_{6n+4}\}$ converges to $\frac{1}{2}$
Q.37 Let
\[ f(x) = \cos(|\pi - x|) + (x - \pi) \sin |x| \text{ and } g(x) = x^2 \text{ for } x \in \mathbb{R}. \]
If \( h(x) = f(g(x)) \), then
(A) \( h \) is not differentiable at \( x = 0 \)
(B) \( h'(\sqrt{\pi}) = 0 \)
(C) \( h''(x) = 0 \) has a solution in \((-\pi, \pi)\)
(D) there exists \( x_0 \in (-\pi, \pi) \) such that \( h(x_0) = x_0 \)

Q.38 Let \( f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R} \) be given by
\[ f(x) = (\sin x)^{\pi} - \pi \sin x + \pi. \]
Then which of the following statements is/are TRUE?
(A) \( f \) is an increasing function
(B) \( f \) is a decreasing function
(C) \( f(x) > 0 \) for all \( x \in \left(0, \frac{\pi}{2}\right) \)
(D) \( f(x) < 0 \) for some \( x \in \left(0, \frac{\pi}{2}\right) \)

Q.39 Let
\[ f(x, y) = \begin{cases} 
\frac{|x|}{|x| + |y|} \sqrt{x^4 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0). 
\end{cases} \]
Then at \((0, 0)\),
(A) \( f \) is continuous
(B) \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} \) does not exist
(C) \( \frac{\partial f}{\partial x} \) does not exist and \( \frac{\partial f}{\partial y} = 0 \)
(D) \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} = 0 \)

Q.40 Let \( \{a_n\} \) be the sequence of real numbers such that
\[ a_1 = 1 \text{ and } a_{n+1} = a_n + a_n^2 \text{ for all } n \geq 1. \]
Then
(A) \( a_4 = a_1(1 + a_1)(1 + a_2)(1 + a_3) \)
(C) \( \lim_{n \to \infty} \frac{1}{a_n} = 1 \)
(B) \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \)
(D) \( \lim_{n \to \infty} a_n = 0 \)
SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q. 41 Let \( x \) be the 100-cycle \((1 \ 2 \ 3 \cdots \ 100)\) and let \( y \) be the transposition \((49 \ 50)\) in the permutation group \( S_{100} \). Then the order of \( xy \) is ______

Q. 42 Let \( W_1 \) and \( W_2 \) be subspaces of the real vector space \( \mathbb{R}^{100} \) defined by

\[
W_1 = \{ (x_1, x_2, \ldots, x_{100}) : x_i = 0 \text{ if } i \text{ is divisible by 4} \},
\]

\[
W_2 = \{ (x_1, x_2, \ldots, x_{100}) : x_i = 0 \text{ if } i \text{ is divisible by 5} \}.
\]

Then the dimension of \( W_1 \cap W_2 \) is _____

Q. 43 Consider the following system of three linear equations in four unknowns \( x_1, x_2, x_3 \) and \( x_4 \)

\[
\begin{align*}
   x_1 + x_2 + x_3 + x_4 &= 4, \\
   x_1 + 2x_2 + 3x_3 + 4x_4 &= 5, \\
   x_1 + 3x_2 + 5x_3 + kx_4 &= 5.
\end{align*}
\]

If the system has no solutions, then \( k = \) ________

Q. 44 Let \( \vec{F}(x, y) = -y \hat{i} + x \hat{j} \) and let \( C \) be the ellipse

\[
\frac{x^2}{16} + \frac{y^2}{9} = 1
\]

oriented counter clockwise. Then the value of \( \oint_C \vec{F} \cdot d\vec{r} \) (round off to 2 decimal places) is____________

Q. 45 The coefficient of \( (x - \frac{\pi}{2}) \) in the Taylor series expansion of the function

\[
f(x) = \begin{cases} 
   4(1 - \sin x), & x \neq \frac{\pi}{2} \\
   \frac{2x - \pi}{2}, & x = \frac{\pi}{2}
\end{cases}
\]

about \( x = \frac{\pi}{2} \), is ____________
Q.46 Let \( f: [0, 1] \rightarrow \mathbb{R} \) be given by

\[
f(x) = \frac{\left(1 + x^3\right)^3 + \left(1 - x^3\right)^3}{8(1 + x)}.
\]

Then

\[
\max \{ f(x): x \in [0,1] \} - \min \{ f(x): x \in [0,1] \}
\]

is _________

Q.47 If

\[
g(x) = \int_{x(x - 2)}^{4x - 5} f(t) \, dt, \text{ where } f(x) = \sqrt{1 + 3x^4} \text{ for } x \in \mathbb{R}
\]

then \( g'(1) = \) ______

Q.48 Let

\[
f(x, y) = \begin{cases} 
\frac{x^3 + y^3}{x^2 - y^2}, & x^2 - y^2 \neq 0 \\
0, & x^2 - y^2 = 0.
\end{cases}
\]

Then the directional derivative of \( f \) at \((0,0)\) in the direction of \( \frac{4}{5} \hat{i} + \frac{3}{5} \hat{j} \) is __________

Q.49 The value of the integral

\[
\int_{-1}^{1} \int_{-1}^{1} |x + y| \, dx \, dy
\]

(round off to 2 decimal places) is __________

Q.50 The volume of the solid bounded by the surfaces \( x = 1 - y^2 \) and \( x = y^2 - 1 \), and the planes \( z = 0 \) and \( z = 2 \) (round off to 2 decimal places) is __________

**Q. 51 – Q. 60 carry two marks each.**

Q.51 The volume of the solid of revolution of the loop of the curve \( y^2 = x^4 (x + 2) \) about the \( x \)-axis (round off to 2 decimal places) is __________
Q.52  The greatest lower bound of the set 
\[ \left\{ (e^n + 2^n)^{\frac{1}{n}} : n \in \mathbb{N} \right\}, \]
(round off to 2 decimal places) is ______________

Q.53  Let \( G = \{ n \in \mathbb{N} : n \leq 55, \ \gcd(n, 55) = 1 \} \) be the group under multiplication modulo 55. Let \( x \in G \) be such that \( x^2 = 26 \) and \( x > 30 \). Then \( x \) is equal to___________

Q.54  The number of critical points of the function 
\[ f(x, y) = (x^2 + 3y^2)e^{-(x^2+y^2)} \]
is __________

Q.55  The number of elements in the set \( \{ x \in S_3; x^4 = e \} \), where \( e \) is the identity element of the permutation group \( S_3 \), is __________

Q.56  If \( \begin{pmatrix} 2 \\ y \\ z \end{pmatrix}, y, z \in \mathbb{R}, \) is an eigenvector corresponding to a real eigenvalue of the matrix \( \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -4 \\ 0 & 1 & 3 \end{pmatrix} \)
then \( z - y \) is equal to___________

Q.57  Let \( M \) and \( N \) be any two \( 4 \times 4 \) matrices with integer entries satisfying 
\[ MN = 2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]
Then the maximum value of \( \det(M) + \det(N) \) is __________

Q.58  Let \( M \) be a \( 3 \times 3 \) matrix with real entries such that \( M^2 = M + 2I \), where \( I \) denotes the \( 3 \times 3 \) identity matrix. If \( \alpha, \beta \) and \( \gamma \) are eigenvalues of \( M \) such that \( \alpha \beta \gamma = -4 \), then \( \alpha + \beta + \gamma \) is equal to___________

Q.59  Let \( y(x) = xv(x) \) be a solution of the differential equation
\[ x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = 0. \]
If \( v(0) = 0 \) and \( v(1) = 1 \), then \( v(-2) \) is equal to___________
Q.60 If $y(x)$ is the solution of the initial value problem

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = 0, \quad y(0) = 2, \quad \frac{dy}{dx}(0) = 0,$$

then $y(ln 2)$ is (round off to 2 decimal places) equal to ____________

END OF THE QUESTION PAPER